Week 10 - Wednesday

COMP 2230

Last time

- Independent events
- Introduction to graphs

Questions?

Assignment 5

Logical warmup

- You work in a pizzeria, and it's a busy night
- You forgot that there was already a pizza in the oven, and you throw a second, identical pizza in
- The huge oven is built for many pizzas, but the new pizza lands partially on top of the first one
- What's the probability that the center of the top pizza is directly over the bottom pizza instead of over the bare oven surface?

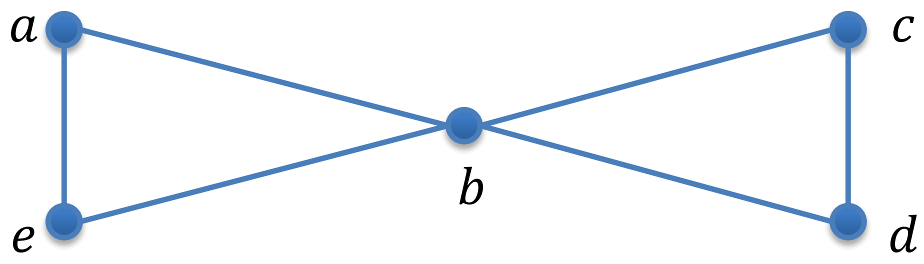
Paths and Circuits

Euler circuits

- What if you want to find an Euler circuit of your own?
- If a graph is connected, non-empty, and every node in the graph has even degree, the graph has an Euler circuit
- Algorithm to find one:
  1. Pick an arbitrary starting vertex
  2. Move to an adjacent vertex and remove the edge you cross from the graph
     - Whenever you choose such a vertex, pick an edge that will not disconnect the graph
  3. If there are still uncrossed edges, go back to Step 2

Hamiltonian circuits

- An Euler circuit has to visit every edge of a graph exactly once
- A **Hamiltonian circuit** must visit every vertex of a graph exactly once (except for the first and the last)
- If a graph G has a Hamiltonian circuit, then G has a subgraph H with the following properties:
  - H contains every vertex of G
  - H is connected
  - H has the same number of edges as vertices
  - Every vertex of H has degree 2
- In some cases, you can use these properties to show that a graph does **not** have a Hamiltonian circuit
- In general, showing that a graph has or does not have a Hamiltonian circuit is NP-complete (widely believed to take exponential time)
- Does the following graph have a Hamiltonian circuit?



Matrix Representations of Graphs and Isomorphism

Three-Sentence Summary

Matrix Representations of Graphs

Matrices

- As you presumably know, a matrix is a rectangular array of elements
- An $m \times n$ matrix has m rows and n columns

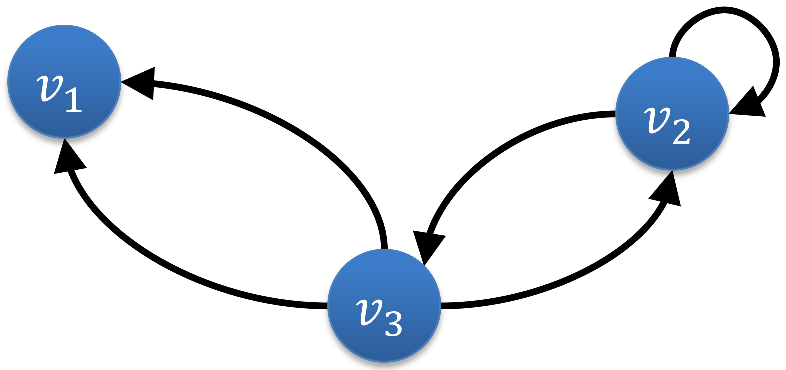
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Graph representations

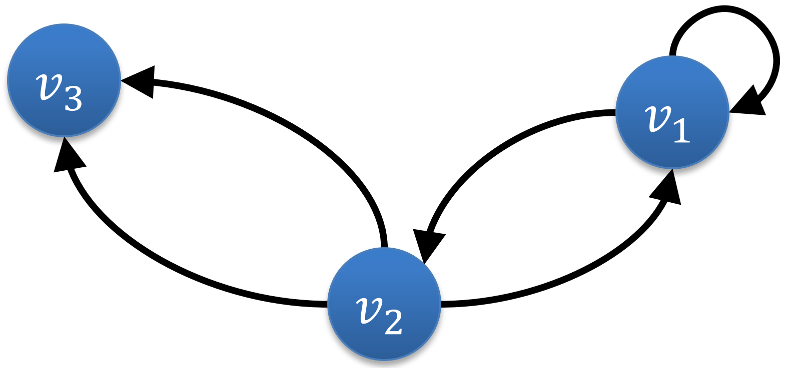
- There are many, many different ways to represent a graph
- If you get tired of drawing pictures or listing ordered pairs, a matrix is not a bad way
- To represent a graph as an **adjacency matrix**, make an $n \times n$ matrix, where n is the number of vertices
- Let the nonnegative integer at a_{ij} give the number of edges from vertex i to vertex j
- A simple graph will always have either 1 or 0 for every location

Graph to matrix examples

- What is the adjacency matrix for the following graph?



- What about for this one?



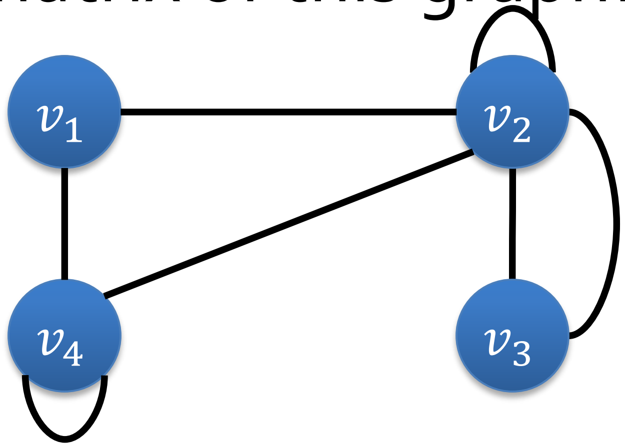
Matrix to graph example

- Draw a graph corresponding to this matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

Another graph to matrix

- What's the adjacency matrix of this graph?



- Note that the matrix is **symmetric**
- In a symmetric matrix, $a_{ij} = a_{ji}$ for all $1 \leq i \leq n$ and $1 \leq j \leq n$
- All undirected graphs have a symmetric matrix representation

Matrix multiplications

- To multiply matrices A and B, it must be the case that A is an $m \times k$ matrix and that B is a $k \times n$ matrix
- Then, the i^{th} row, j^{th} column of the result is the dot product of the i^{th} row of A with the j^{th} column of B
- In other words, we could compute element c_{ij} in the result matrix C as follows:

$$\sum_{x=1}^{k} a_{ix} \cdot b_{xj}$$

Matrix multiplication practice

- Multiply matrices A and B

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 3 \\ 2 & 2 \\ -2 & -1 \end{bmatrix}$$

A few points about matrix multiplication

- Matrix multiplication is associative
  - That is, $A(BC) = (AB)C$
- Matrix multiplication is **not** commutative
  - AB is not always equal to BA (for one thing, BA might not even be legal if AB is)
- There is an $n \times n$ identity matrix I such that, for any $m \times n$ matrix A, $AI = A$
  - I is all zeroes, except for the diagonal (where row = column) which is all ones
- We can raise square matrices to powers using the following recursive definition
  - $A^0 = I$, where I is the $n \times n$ identity matrix
  - $A^k = AA^{k-1}$, for all integers $k \geq 1$

Finding powers of a matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

- Is A symmetric?
- Compute A^0
- Compute A^1
- Compute A^2
- Compute A^3

Matrix powers for graphs

- We can find the number of walks of length k that connect two vertices in a graph by raising the adjacency matrix of the graph to the k^{th} power
- Raising a matrix to the 0^{th} power means you can only get from a vertex to itself (identity matrix)
- Raising a matrix to the first power means that the number of paths of length one from one vertex to another is exactly the number of edges between them
- The result holds for all k, but we aren't going to prove it

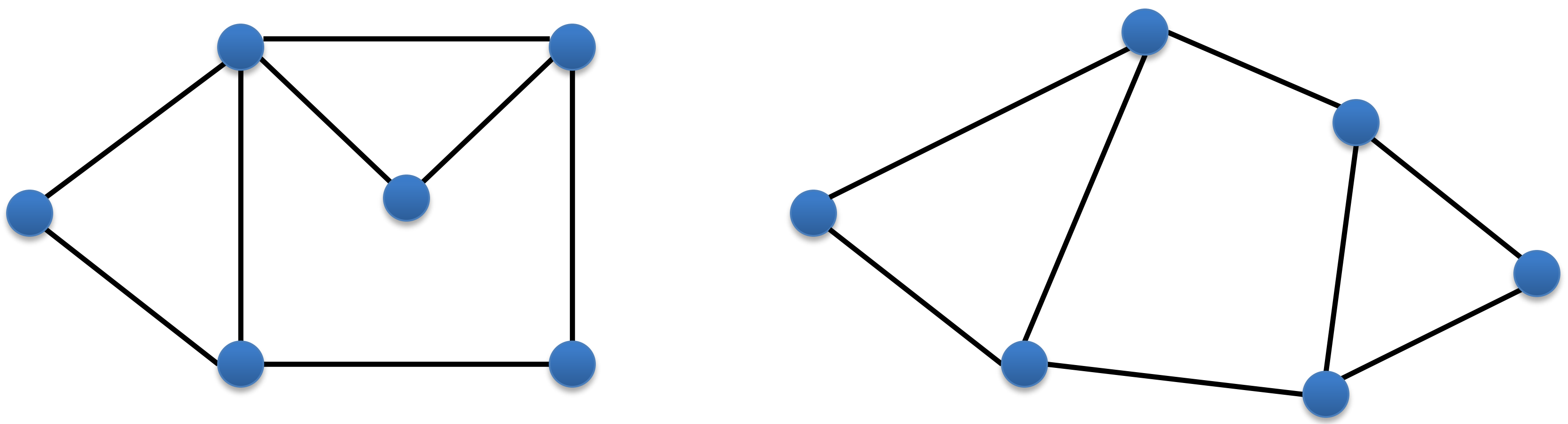
Graph Isomorphism

Isomorphism invariants

- A property is called an **isomorphism invariant** if its truth or falsehood does not change when considering a different (but isomorphic) graph
- Ten common isomorphism invariants:
  1. Has n vertices
  2. Has m edges
  3. Has a vertex of degree k
  4. Has m vertices of degree k
  5. Has a circuit of length k
  6. Has a simple circuit of length k
  7. Has m simple circuits of length k
  8. Is connected
  9. Has an Euler circuit
  10. Has a Hamiltonian circuit

Using invariants to disprove isomorphism

- If any of the invariants have different values for two different graphs, those graphs are not isomorphic
- Use the ten invariants given to show that the following pair of graphs is not isomorphic



Proving isomorphism

- To **prove** that two graphs are isomorphic, mathematicians have to relabel the nodes on one graph to be the same as the other
- To **disprove** that two graphs are isomorphic, mathematicians should show that some invariant doesn't hold
  - For example, two graphs with different numbers of edges can't be isomorphic
- The processes are asymmetrical
- Algorithms:
  - The best known algorithm to see if two graphs were isomorphic ran in $O\left(2^{\sqrt{n \log n}}\right)$
  - In November 2015, someone claimed to find an algorithm that runs in $O\left(2^{\log^3 n}\right)$, also called quasipolynomial time, but the full algorithm hasn't been published

Upcoming

Next time…

- Trees
- Rooted trees
- Spanning trees and shortest paths

Reminders

- **Work on Assignment 5**
- Read 10.4, 10.5, and and 10.6
  - Prepare a three-sentence summary
  - Extra credit if you get called on